Income and inequality under asymptotically full automation

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Abstract

We study a minimal model of automation in which labor is asymptotically replaced by capital in all tasks. Heterogeneity in broadly defined financial frictions naturally produces "workers" and "capitalists." If tasks are gross complements and automation progresses sufficiently slowly, asymptotically full automation does not immiserate workers despite endogenously shrinking hours worked. Faster automation, however, lowers workers' consumption growth and output share. The threshold speed of automation separating these outcomes maps to observables, and a first-pass calibration suggests the current speed is below this threshold. For faster automation, redistribution funded by capital taxation serves workers better than deliberately slowing automation.

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1 Introduction

Technological advance has allowed the automation of many tasks historically performed by workers. Recent progress in artificial intelligence raises the prospect that, asymptotically, *all* tasks will be automated, prompting concerns about how future individuals will earn their living. Will asymptotic automation generate widely shared economic benefits, as many past technological advances have done? Or will it instead lead to widespread immiseration, with its benefits flowing overwhelmingly to the owners of capital?

In this paper, we analytically characterize the implications of standard economic forces for this question. We establish that whether or not the benefits of automation are broadly shared hinges on whether the speed of automation is above or a below a critical threshold, which we characterize and relate to observables.

For automation speeds below the threshold, the Baumol and Bowen (1965) force implies that wages for work on yet-to-be-automated tasks rise quickly enough to offset the shrinking set of such tasks.[1] Workers' income rises in line with overall GDP, even as workers enjoy increasing leisure. Both workers' income and GDP grow faster as the rate of automation locally increases. While locally faster automation shrinks the labor share of the economy, workers' income nonetheless increases in absolute terms, because of benefits of a higher growth rate.

In contrast, automation speeds above the threshold are threatening for workers. The specific outcome depends on parameter values, and in particular, the strength of complementarities between consumption and leisure, and across different tasks. For weak complementarities, the growth rate of workers' consumption is locally decreasing in the speed of automation. Moreover, workers' consumption is an asymptotically negligible share of the economy. For strong complementarities, multiple equilibria co-exist—one in which workers share in the fruits of automation, as described above, and a second in which workers' consumption and income grow at strictly lower rates, both relative to the first equilibrium and relative to the growth rate of the economy.

As noted, we relate the critical threshold rate of automation to observables. A first-pass calibration suggests that the current speed of automation in the US is below this threshold, and accordingly, that the current path of automation will result in the benefits of technological advance being widely shared. But the existence of a threshold rate also highlights why the prospect of increasing rates of automation should be taken seriously.

Workers and capitalists emerge as distinct groups in our analysis solely due to differences in their investment abilities. That is, workers are simply agents who are worse at investing capital. Heterogeneity of investment returns across individuals is well-documented (e.g., Fagereng et al., 2020; Bach et al., 2020; Smith et al., 2022). We model this heterogeneity as stemming directly from financial frictions: Workers experience a higher investment cost than capitalists. We note that this is isomorphic to differences in time preference rates, which are often associated with wealth inequality

[1] See Aghion et al. (2019) for a related observation in a representative agent economy with exogenous labor supply and savings rates.

(e.g., Ramsey, 1928; Krusell and Smith, 1998) and enjoy similar empirical support (e.g., Lawrance, 1991; Epper et al., 2020), and that heterogeneity in risk aversion (e.g., Cronqvist and Siegel, 2015), background risk, or ability to diversify wealth would generate similar effects. Throughout, we refer to the heterogeneity in investment returns as *financial frictions*, though this term should be interpreted broadly.

To keep our analysis as close to a benchmark as possible, we assume that "workers" and "capitalists" are ex ante identical in all other respects—in particular, they have identical preferences and labor productivities.

The most striking effect of financial frictions on equilibrium outcomes emerges when both the complementarities of different tasks and of consumption and leisure are strong. In this case, an increase in financial frictions makes capital dominance more likely. Roughly speaking, greater financial frictions push workers to save more to offset these frictions, and to reduce both consumption and leisure (the two are complements). The associated increase in labor depresses wages, pushing towards capital dominance. Consequently, government policies to reduce financial frictions help workers both directly, and via the equilibrium effect of whether capital dominance occurs.

Our analysis suggests that workers respond to automation by reducing hours worked. Although one may be tempted to conclude that shrinking labor supply contributes to capital dominance, the reverse is true. In a perturbation of our model in which labor supply of workers is exogenous and constant, wage growth is slower and capital dominance occurs for a wider range of parameter values.

Our aim in this paper is to take seriously the prospect that all tasks will eventually be automated, and to analyze the consequences for the economy. As such, we take the speed of automation as exogenous; the key endogenous objects are workers' and capitalists' consumption and labor responses, and the associated equilibrium wages and capital return rates.

Some of the results above depend on consumption and leisure being (gross) complements. The opposite assumption of (gross) substitutes implies—for many parameter values—increasing labor over time, a prediction at odds with time-series and cross-country evidence (e.g., Becker, 1965; Huberman and Minns, 2007; Feenstra et al., 2015).[2] Bick et al. (2018) further find that for most countries, the amount of hours worked is decreasing in the wage.

Related literature:

Our conceptualization of the automation process directly follows the insightful work of Aghion et al. (2019). Relative to that paper, we endogenize both savings and labor supply, and depart from the representative agent framework by introducing financial frictions which in turn generate distinct populations of workers and capitalists. These features are necessary to address the questions laid out above and to take the fully characterized capital-dominance threshold to the data.

Following earlier models of automation, we view technological progress as a gradual replacement

[2] Keynes (1930) famously predicted a 15-hour workweek for his grandchildren thanks to rising productivity. Boppart and Krusell (2020) write: "As it turned out, Keynes was wildly off quantitatively, but he was right qualitatively (on this issue)."

of labor with capital as a production factor (Zeira, 1998; Acemoglu and Restrepo, 2018a,b). In this view, automation extends beyond artificial intelligence to major sources of economic growth since the Industrial Revolution.

Acemoglu and Restrepo (2018b) endogenize automation and the invention of new tasks in which labor has a comparative advantage. They find that long-run factor shares are stable if the long-run rental rate of capital is sufficiently high. Intuitively, automation reduces the cost of labor, thereby discouraging further automation and encouraging the development of new tasks. In our case, the labor share stabilizes despite exogenous automation of all tasks in the limit. Our mechanism works through complementarity between tasks (the Baumol effect) and between consumption and leisure.

In Moll et al. (2022), automation increases inequality via returns to wealth and by facilitating stagnant wages. As in their model, returns to capital rise with the speed of automation (up to the capital-dominance threshold), and capital income tends to generate inequality in consumption growth and consumption shares. Unlike in their model, where inequality is driven by stochastic capital accumulation, our result is obtained from financial frictions motivated by the empirically observed differences in average returns to capital. We also find that automation *can* reduce wages, but only does so under very specific circumstances. With a low-to-medium speed of automation, wage growth increases with the rate of automation.

Our analysis is predominantly concerned with the limit of full automation and the asymptotic factor shares. Nonetheless, the key forces driving our mechanism also speak to three long-run empirical trends: (i) the decline in hours worked (Boppart and Krusell, 2020), (ii) the rising capital share (Karabarbounis and Neiman, 2014), which our model naturally connects to a decline in TFP growth (Philippon, 2023), and (iii) a reallocation in output shares towards services (Boppart, 2014). We discuss these trends in the context of the model in Subsection 5.6.

Our formulation of the capital-labor complementarity is distinct from the literature explaining changes in skill premia through skill-biased technological change, that is, capital-skill complementarity in production (e.g., Acemoglu, 1998; Krusell et al., 2000; Autor et al., 2003).[3] Instead, we do not take a stance on the types of tasks that remain unautomated for longer, meaning the wages from those tasks may be earned by nurses, teachers, athletes, or—as Baumol and Bowen (1965) would have it—performing artists.

Because the rate of automation in our analysis is exogenously constant, the growth rate of the economy converges in the long-run. In this sense, our analysis does not generate a "singularity" in which the growth rate accelerates over time (see Nordhaus (2021), and references therein). It does, however, suggest that if, for whatever reason, the rate of automation climbs sufficiently high it overwhelms the economic forces stemming from the complementarity of different tasks, and the complementarity of consumption and leisure, and the labor share of the economy converges to zero.

[3] Guerreiro et al. (2021) and Ray and Mookherjee (2022) study settings in which it is technologically possible to automate all routine tasks immediately, and only the cost of automation ("robots") prevents this from happening. Instead, a crucial assumption for the Baumol-force to operate is that at any finite time some tasks cannot be automated, though the number of such tasks asymptotes to zero.

2 Model

2.1 Preferences and endowments

There is a unit mass of infinitely lived economic agents, each of whom continuously consumes, works, and adjusts capital holdings. Population growth equals 0 (we have verified that, as in the standard neoclassical growth model, steady state outcomes are independent of population growth). Each agent discounts the future at rate ρ. There is no uncertainty.

Agents are either "workers" or "capitalists" (denoted by subscript 'o' for "owners") with respective measures λ_w and λ_o. The only difference between the two groups is that capitalists are more effective at holding capital. Let $K_{i,t}$, $L_{i,t}$, and $C_{i,t}$ respectively denote the date t capital holding, time spent working, and consumption of an agent of type $i = w, o$. Moreover, let W_t and R_t denote the date t wage rate and return on capital (not including depreciation and other holding costs). Capital accumulation for a type-i agent is

$$\dot{K}_{i,t} = R_t K_{i,t} + W_t L_{i,t} - \delta_i K_{i,t} - C_{i,t},$$

where δ_i is the combined depreciation and holding costs experienced by type-i agents. We assume throughout that[4]

$$\delta_w > \delta_o.$$

Each agent's flow endowment of time is 1, so that flow leisure is $1 - L_{i,t}$. Regardless of type, each agent's flow utility is

$$\frac{1}{1-\gamma}\left(C_{i,t}^{\frac{\eta-1}{\eta}} + \omega\left(1 - L_{i,t}\right)^{\frac{\eta-1}{\eta}}\right)^{\frac{1-\gamma}{1-\frac{1}{\eta}}}.$$

Here, ω is a parameter determining the relative importance of leisure versus consumption; η is the elasticity of substitution between consumption and leisure; while γ is the standard coefficient from power utility functions. In the special case of $\omega = 0$, the intertemporal elasticity of substitution is $\frac{1}{\gamma}$.

Agents are credit constrained, in the sense that capital holdings cannot be too negative. For simplicity, we set the credit constraint at 0, i.e., $K_{i,t}$ must satisfy

$$K_{i,t} \geq 0.$$

We impose the standard transversality condition

$$\lim_{t\to\infty} K_{i,t} \int_0^t e^{-(R_s - \delta_i)} ds = 0. \quad (1)$$

[4]See Appendix C for an analysis of the (easier) representative agent case.

2.2 Technology

Following existing literature (see introduction), we conceptualize output as a single, composite consumption good that is composed of a unit measure of complementary "tasks," with the elasticity of substitution across any pair of tasks equal to σ. A "task" should be interpreted generally. In contrast to Acemoglu and Restrepo (2018b), we think of tasks as being fundamental "needs" such as food, shelter, entertainment, transport etc., so that the set of tasks remains fixed over time (see also Aghion et al. (2019)).

Importantly, we assume that tasks are gross complements, i.e., $\sigma < 1$. It is this assumption that allows the Baumol force to potentially operate. Given our output formulation, the interpretation of $\sigma < 1$ nests both preference-based complementarity across different consumption goods and technology-based complementarity in production processes that combine intermediate tasks into ultimate output goods.

Let α_t be the fraction of tasks that has been automated at date t. Non-automated tasks are executed using only labor. For automated tasks, capital and labor are perfect substitutes. In equilibrium, capital grows without bound, so capital becomes abundant relative to labor; consequently, in equilibrium automated tasks are (eventually) executed using only capital.

Let K_t and L_t denote aggregate capital and labor:

$$\begin{aligned} K_t &= \lambda_w K_{w,t} + \lambda_o K_{o,t} \\ L_t &= \lambda_w L_{w,t} + \lambda_o L_{o,t}. \end{aligned}$$

Hence date t output is

$$\begin{aligned} F_t = F\left(K_t, L_t; \alpha_t\right) &= \left(\alpha_t \left(A_K \frac{K_t}{\alpha_t}\right)^{\frac{\sigma-1}{\sigma}} + (1-\alpha_t)\left(A_L \frac{L_t}{1-\alpha_t}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(\alpha_t^{\frac{1}{\sigma}} \left(A_K K_t\right)^{\frac{\sigma-1}{\sigma}} + (1-\alpha_t)^{\frac{1}{\sigma}} \left(A_L L_t\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}. \end{aligned} \tag{2}$$

That is: each of the α_t automated tasks receives capital $\frac{K_t}{\alpha_t}$, and each of the $1-\alpha_t$ non-automated tasks receives labor $\frac{L}{1-\alpha_t}$. Parameters A_K and A_L determine the productivity of capital and labor.

For calibration (Section 6), note that the elasticity of substitution across tasks, σ, coincides with the elasticity of substitution between capital and labor, which is estimated by a sizeable literature.

The marginal products of capital and labor are

$$F_{K,t} = \frac{\partial}{\partial K_t} F\left(K_t, L_t; \alpha_t\right) = \alpha_t^{\frac{1}{\sigma}} A_K^{\frac{\sigma-1}{\sigma}} K_t^{-\frac{1}{\sigma}} F_t^{\frac{1}{\sigma}} \tag{3}$$

$$F_{L,t} = \frac{\partial}{\partial L_t} F\left(K_t, L_t; \alpha_t\right) = (1-\alpha_t)^{\frac{1}{\sigma}} A_L^{\frac{\sigma-1}{\sigma}} L_t^{-\frac{1}{\sigma}} F_t^{\frac{1}{\sigma}}. \tag{4}$$

As time passes, more and more tasks are automated. Our focus is on the consequences of automation, so we take the automation process as exogenous, reflecting an immutable "march of progress." Specifically, automation proceeds at rate $\theta > 0$:

$$\dot{\alpha}_t = (1 - \alpha_t)\,\theta.$$

Asymptotically all tasks are automated; but at any finite time t, some tasks remain non-automated.

Looking ahead, faster automation is associated with lower relative wages in equilibrium, i.e., lower $\frac{F_{L,t}}{F_{K,t}}$. Consequently, the likely effects of endogenizing the pace of automation hinge on whether innovation is labor- or capital-intensive. Labor-intensive automation would amplify the effects of exogenous variation in automation rates. In contrast, capital-intensive automation—often invoked in "singularity" discussions—would dampen the effects of exogenous variation.

2.3 Equilibrium

An equilibrium consists of paths $\{K_{i,t}, C_{i,t}, L_{i,t}\}$ for $i = w, o$ and rental rates and wages $\{R_t, W_t\}$ such that $\{K_{i,t}, C_{i,t}, L_{i,t}\}$ is individually optimal for each agent given the path of $\{R_t, W_t\}$, while rental rates and wages are determined by the competitive conditions

$$\begin{aligned} R_t &= F_K\left(K_t, L_t; \alpha_t\right) \\ W_t &= F_L\left(K_t, L_t; \alpha_t\right). \end{aligned}$$

2.4 Parameter assumptions

To capture the Baumol effect, and consistent with empirical estimates, we assume that tasks are gross complements,

$$\sigma < 1.$$

We further assume that consumption and leisure are gross complements,

$$\eta < 1.$$

Under the alternative assumption ($\eta > 1$), many parameter configurations deliver equilibria in which leisure converges to 0, while observed trends indicate increases in leisure. These trends also motivate our deviation from KPR-preferences (King et al., 1988) which generate stable labor supply.

Throughout, we assume that for $i = o, w$

$$A_K - \delta_i > \rho > (1 - \gamma)\left(A_K - \delta_i\right). \tag{5}$$

The first inequality ensures capital growth in a benchmark economy with production $A_K K_t$, while the second inequality ensures the transversality condition is satisfied in the same benchmark.

3 Labor share dynamics and capital dominance

We first define our notions of a stable labor share and capital dominance; note some important implications of these definitions; and derive laws of motion for key aggregate quantities. This section uses only the definition of the production function (2).

The date-t labor share of the economy is

$$X_t \equiv \frac{L_t F_{L,t}}{F_t} = 1 - \frac{K_t F_{K,t}}{F_t}. \tag{6}$$

Definition 1 *We say that capital dominance occurs if* $\lim_{t\to\infty} X_t = 0$*. If instead* $\lim_{t\to\infty} X_t > 0$*, we say that the labor share is stable.*

Throughout, we write lim for $\lim_{t\to\infty}$, and typically omit time subscripts when characterizing asymptotic behavior.

Let g_R and g_W denote the growth rates of return-on-capital R_t and wages W_t, with parallel notation for growth rates of other quantities. From (3) and (4),

$$g_{R,t} = \frac{1}{\sigma}\left(g_{F,t} - g_{K,t} + \theta\frac{1-\alpha_t}{\alpha_t}\right), \tag{7}$$

$$g_{W,t} = \frac{1}{\sigma}\left(g_{F,t} - g_{L,t} - \theta\right), \tag{8}$$

Hence capital and labor shares $1-X$ and X evolve according to

$$g_{1-X,t} = g_{K,t} + g_{R,t} - g_{F,t} = (1-\sigma)\, g_{R,t} + \theta\frac{1-\alpha_t}{\alpha_t}. \tag{9}$$

$$g_{X,t} = g_{L,t} + g_{W,t} - g_{F,t} = (1-\sigma)\, g_{W,t} - \theta. \tag{10}$$

So capital dominance occurs if

$$\lim g_W < \frac{\theta}{1-\sigma},$$

while a stable labor share requires

$$\lim g_W = \frac{\theta}{1-\sigma}. \tag{11}$$

Lemma 1 *Output evolves according to*

$$g_{F,t} = (1-X_t)\, g_{K,t} + X_t g_{L,t} + \frac{\theta}{1-\sigma}\left(1 - \frac{1-X_t}{\alpha_t}\right). \tag{12}$$

We characterize the economy as the fraction of automated tasks α_t approaches 100%. We focus on equilibria in which the capital share has a well-defined and strictly positive limit. From (7) and (9) it is immediate that, asymptotically, output and capital grow at the same rate:

$$\lim g_F = \lim g_K. \tag{13}$$

From (7), the rental rate $F_{K,t}$ asymptotically converges; define

$$\bar{F}_K \equiv \lim F_K.$$

The capital share is straightforwardly a function of the rental rate:

$$1 - X_t = \alpha_t \left(\frac{F_{K,t}}{A_K}\right)^{1-\sigma}, \tag{14}$$

and so in particular the limiting capital share is

$$\lim (1 - X) = \left(\frac{\bar{F}_K}{A_K}\right)^{1-\sigma}. \tag{15}$$

In a capital-dominant equilibrium,

$$\lim \frac{F}{K} = \bar{F}_K = A_K. \tag{16}$$

Finally, the bounded nature of labor L_i and leisure $1 - L_i$ means that, provided labor has a well-defined asymptotic value, the asymptotic growth rates of leisure and labor are weakly negative:

$$\begin{aligned} \lim g_{1-L} &\leq 0 & (17)\\ \lim g_L &\leq 0. & (18) \end{aligned}$$

Moreover, at least one of (17) and (18) holds with equality.

4 Analysis

4.1 Optimality conditions

The marginal utilities of consumption and leisure are

$$\begin{aligned} MU_{C_i,t} &= C_{i,t}^{-\frac{1}{\eta}} \left(C_{i,t}^{\frac{\eta-1}{\eta}} + \omega (1 - L_{i,t})^{\frac{\eta-1}{\eta}}\right)^{\frac{1-\eta\gamma}{\eta-1}} \\ MU_{1-L_i,t} &= \omega (1 - L_{i,t})^{-\frac{1}{\eta}} \left(C_t^{\frac{\eta-1}{\eta}} + \omega (1 - L_{i,t})^{\frac{\eta-1}{\eta}}\right)^{\frac{1-\eta\gamma}{\eta-1}}. \end{aligned}$$

The intratemporal and intertemporal optimality conditions are

$$\begin{aligned} W_t C_{i,t}^{-\frac{1}{\eta}} &\leq \omega (1 - L_{i,t})^{-\frac{1}{\eta}}, & (19)\\ \frac{\partial}{\partial t} \ln MU_{C_i,t} &\leq -(R_t - \delta_i - \rho), & (20) \end{aligned}$$

with (19) at equality if labor is strictly positive ($L_{i,t} > 0$), and (20) at equality if capital-holding is strictly positive ($K_{i,t} > 0$).

Looking ahead: The fact that workers and capitalists differ in δ_i makes the corners of no-work and no-capital relevant.

If type-i agents work then, from (19),

$$g_{C_i} - g_{1-L_i} = \eta g_W; \tag{21}$$

and,

$$MU_{C,t} = C_t^{-\gamma} \left(1 + \omega^\eta W_t^{1-\eta}\right)^{\frac{1-\eta\gamma}{\eta-1}}. \tag{22}$$

Consequently, if type-i agents work their marginal utility grows according to

$$\frac{\partial}{\partial t} \ln MU_{C_i} = -\gamma g_{C_i} - (1-\eta\gamma) \frac{\omega^\eta g_W}{W^{\eta-1} + \omega^\eta},$$

while if they are at the no-work corner,

$$\frac{\partial}{\partial t} \ln MU_{C_i} = -\frac{1}{\eta} g_{C_i} + \frac{1-\eta\gamma}{\eta} \frac{g_{C_i} C_i^{\frac{\eta-1}{\eta}}}{C_i^{\frac{\eta-1}{\eta}} + \omega} = -g_{C_i} \frac{\gamma C_i^{\frac{\eta-1}{\eta}} + \frac{\omega}{\eta}}{C_i^{\frac{\eta-1}{\eta}} + \omega}. \tag{23}$$

The assumption that capital is sufficiently productive to drive long-run growth (5), together with the complementarity of consumption and leisure ($\eta < 1$), ensure that in all equilibria:

Lemma 2 *Asymptotically, the leisure growth rate of both groups is* 0, $\lim g_{1-L_i} = 0$*; and wages and consumption of both groups grow at a strictly positive rate.*

4.2 Factor segmentation

Because asymptotic leisure growth is 0, if both workers and capitalists work asymptotically then their consumption growth rates would coincide, by (21). In this case, it is impossible to satisfy the intertemporal optimality condition (20) with equality for both groups. Consequently:

Corollary 1 *At least one group must be either at the no-capital corner or the no-labor corner.*

By Lemma 2, consumption grows without bound for both groups, as does the wage rate. From (22) and (23), it follows that regardless of whether or not a group $i = o, w$ works

$$\lim \frac{\partial}{\partial t} \ln MU_{C,i} = -\frac{1}{\eta} \lim g_{C_i}, \tag{24}$$

and the asymptotic intertemporal condition is

$$\lim g_{C_i} \geq \eta \left(\bar{F}_K - \delta_i - \rho\right), \tag{25}$$

with equality for any group that holds capital.

Our next result characterizes which of the no-work and no-capital corners are relevant. It also justifies our terminology of "workers" and "capitalists."

Lemma 3 *Capitalists hold capital and workers work. In a capital-dominant equilibrium, capitalists do not work. In a stable labor share equilibrium, workers do not hold capital.*

Since workers work, and their leisure asymptotes to its upper bound, the fact that both workers and capitalists' labor choices satisfy intratemporal optimality ((19) and (21)) implies:

Corollary 2 *Asymptotically, wages grow strictly faster than workers' consumption,*

$$\lim g_W = \frac{1}{\eta} \lim g_{C_w}; \tag{26}$$

and capitalists' consumption grows weakly faster than workers' consumption:

$$\lim g_{C_o} \geq \lim g_{C_w}.$$

Since capitalists' are advantaged in holding capital, and since their consumption grows at least as fast as that of workers:

Lemma 4 *Asymptotically, output, capitalists' consumption, and capitalists' capital-holdings all grow at the same rate,*

$$\lim g_F = \lim g_{C_o} = \lim g_{K_o}.$$

4.3 Capital-dominant equilibria

From Lemma 3, in a capital-dominant equilibrium capitalists do not work, and hence workers must do so. One possibility is that both capitalists and workers hold capital:

Proposition 1 *A capital-dominant equilibrium in which workers hold capital exists if*

$$\theta \geq (1-\sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o). \tag{27}$$

Consumption growth of group i satisfies

$$\lim g_{C_i} = \eta(A_K - \delta_i - \rho). \tag{28}$$

Labor converges towards 0 according to

$$\lim g_{L_w} = (\eta - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) - \theta.$$

The second possibility is that workers do not hold capital:

Proposition 2 *A capital-dominant equilibrium in which workers do not hold capital exists if complementarities are weak,*[5] $\sigma + \eta > 1$, *and*

$$\theta \in \left[(1-\sigma)\left(A_K - \delta_o - \rho\right), (1-\sigma)\left(A_K - \delta_w - \rho\right) + \eta\left(\delta_w - \delta_o\right)\right]. \tag{29}$$

Capitalists' consumption growth satisfies (28), while workers' consumption growth satisfies

$$\lim g_{C_w} = \eta \frac{\lim g_{C_o} - \theta}{\sigma + \eta - 1} < \lim g_{C_o}. \tag{30}$$

Labor converges towards 0 *according to*

$$\lim g_{L_w} = \frac{\eta - 1}{\eta} \lim g_{C_w}. \tag{31}$$

From Propositions 1 and 2, capital dominance emerges when the rate of automation is sufficiently high.[6] In particular, the Baumol effect, arising from task-complementarity ($\sigma < 1$) pushes against capital dominance. Capital accumulation, which is asymptotically proportional to $A_K - \delta_o - \rho$ in a capital-dominant equilibrium, likewise pushes *against* capital dominance because it increases wages relative to the return on capital. Capital dominance emerges when automation advances sufficiently rapidly relative to the extent of complementarity and the rate of capital accumulation.

Capital dominance is associated with the immiseration of workers *relative* to capitalists. This is immediate if workers do not hold capital (Proposition 2). But even when workers hold capital, they are disadvantaged relative to capitalists in doing so ($\delta_w < \delta_o$). At the same time: Even in a capital-dominant equilibrium workers' consumption grows without bound, even as their leisure approaches its upper bound.

4.4 Stable labor share equilibria

If instead automation proceeds more slowly, a stable labor share emerges. In this case, workers' and capitalists' consumption grow at the same rate.

Proposition 3 *A stable labor share equilibrium exists if*

$$\theta < (1-\sigma)\left(A_K - \delta_o - \rho\right). \tag{32}$$

[5]If complementarities are strong ($\sigma + \eta < 1$) then an unstable equilibrium in which workers do not hold capital exists for automation speeds above the threshold (32) but below the threshold (27). This equilibrium is unstable because a drop in worker consumption is self-reinforcing; see the heuristic argument for the existence of multiple equilibria that follows Proposition 3, though in this case the argument is precise because workers do not hold capital and so optimize period-by-period.

[6]See footnote 8 of Aghion et al. (2019) for a related statement in a representative-agent model with exogenous capital accumulation and labor supply.

Consumption of capitalists and workers grows at same rate,

$$\lim g_{C_o} = \lim g_{C_w} = \frac{\eta\theta}{1-\sigma}. \tag{33}$$

Labor converges towards 0 *according to* (31)*. The labor share converges towards*

$$\lim X = 1 - \left(\frac{\delta_o + \rho + \frac{\theta}{1-\sigma}}{A_K} \right)^{1-\sigma}. \tag{34}$$

4.5 Summary

Together, Propositions 1-3 span the parameter space, and are illustrated by Figure 1. When the combined complementarity of tasks (σ) and of consumption and leisure (η) is strong, stable labor share and capital-dominant equilibria coexist for some parameters (Propositions 1 and 3). Heuristically, coexistence arises because capital dominance is associated with lower worker consumption,[7] which is in turn self-reinforcing: complementarity of consumption and leisure implies that low consumption is associated with workers supplying a lot of labor; and this association is especially strong when η is low. The large quantity of labor supplied is in turn associated with low wages. Because task complementarity is strong, the net effect of more labor but lower wages is lower labor income—which in turn leads to low worker consumption.

5 Discussion

5.1 Will the benefits of automation be widely shared?

Propositions 1-3 address the question of whether the benefits of automation will be widely shared. When automation (θ) is sufficiently slow, and the task complementarities (σ) are sufficiently strong, the answer is yes. Even as all tasks are asymptotically automated, workers' share of the economy remains stable, measured either by income or consumption. Moreover, local increases in the speed of automation benefit workers by increasing their consumption growth, and by the same amount as GDP growth.

In contrast, automation speeds above a threshold threaten workers. In the case of weak complementarities, workers' consumption growth is decreasing in the rate of automation—and strictly so for an interval of automation speeds. Income and consumption inequality both explode, with workers' share of the economy asymptoting to zero. More positively: workers' consumption growth nonetheless remains positive, and as time passes they work vanishingly little, so even in this case workers' absolute living standards improve as more and more tasks are automated.

[7] Formally, Propositions 1 and 3 involve growth rates rather than levels. But here we give a heuristic argument.

For strong complementarities, multiple equilibria exist—one with a stable labor share, and one with capital dominance. Workers' consumption growth is strictly higher in the former, even though both GDP and capitalists' consumption grows strictly faster in the latter:

Lemma 5 *If stable labor share and capital dominant equilibria coexist then workers' consumption grows strictly faster, their labor shrinks to zero strictly faster, and capitalists' consumption grows strictly more slowly,*[8] *in the stable labor share equilibrium than in the capital-dominant equilibrium.*

5.2 Workers' income under capital dominance, and the effect of financial frictions

Capital-dominant equilibria, which by definition feature a vanishing labor share, raise the question of how workers obtain income to consume. The answer depends on the strength of complementarities; the speed of automation; and the strength of financial frictions, broadly defined, and as measured by δ_w.

Consider first the case of strong complementarities ($\sigma + \eta < 1$). In this case, workers hold capital in any (stable) capital-dominant equilibrium (Propositions 1 and 2). Moreover, their consumption is asymptotically entirely funded by capital income:

Corollary 3 *In any equilibrium in which workers hold capital, workers' capital income grows strictly faster than their labor income.*

Because workers hold capital to protect themselves from the consequences of automation in capital-dominant equilibria, an increase in δ_w naturally reduces the growth rate of their consumption (Proposition 1).

Moreover, workers are potentially further harmed by an increase in δ_w because it expands the range of automation speeds for which a capital dominant equilibrium coexists with a stable labor share one (Lemma 5). In this case, small increases in financial frictions potentially cause large drops in workers' consumption growth, highlighting the importance of the efficiency of the financial system, and (depending on interpretation of the origins of $\delta_w - \delta_o$) financial literacy.

Next, consider the case of weak complementarities. When automation is fast enough to deliver capital dominance, there are two subcases to consider. If the rate of automation is very high then labor income falls so quickly that workers again hold capital to protect themselves; by Corollary 3, their consumption is asymptotically entirely funded by capital income. Conversely, if the rate of automation is more moderate then workers do not hold capital, and fund consumption entirely from labor income. Even though the labor share of the economy shrinks, labor income nonetheless grows in absolute terms, enabling strictly positive consumption growth without capital income.

When complementarities are weak, an increase in δ_w is again bad for workers:

[8] Capitalists' consumption grows at the same rate in the two equilibria if (32) holds with equality.

Corollary 4 *An increase in financial frictions from δ_w to $\tilde{\delta}_w > \delta_w$ strictly reduces workers' consumption growth if workers hold capital at the initial value δ_w, and has no effect otherwise.*

5.3 Policy: Automation retardation vs capital taxation

Consider the case in which automation is sufficiently fast that a capital-dominant equilibrium arises. What can a government that wishes to avoid the (relative) immiseration of workers do? We consider the efficacy of a capital tax rebated to workers against a hypothetical benchmark, in which the government can directly lower the exogenous pace of automation, θ. For conciseness, we focus on the case of weak complementarities, and in which capitalists are a small fraction of the population ($\lambda_o \approx 0$). Let $\mathcal{X}$ be the worker-consumption share that the policies target.

Taxation of capital is equivalent to a level-increase in δ_o and δ_w. By itself, such an increase pushes the economy towards a capital-dominant outcome (again, see the threshold condition (32)). The reason is that capital-taxation reduces the growth rate of capital, making it scarcer, and raising its equilibrium return. As discussed following Proposition 2, the net effect is to promote capital dominance (given task-complementarity $\sigma < 1$).

Nonetheless, capital-taxation generates income that a government can redistribute. Given the assumption that the population-share of capitalists is small, a worker-consumption share of $\mathcal{X}$ is funded with a capital tax of $\mathcal{X} A_K$. From Propositions 1 and 2, the associated consumption growth rate is

$$\eta \left(A_K \left(1 - \mathcal{X}\right) - \delta_o - \rho \right). \tag{35}$$

Under the alternative policy, the government lowers θ below the threshold in (32). From Proposition 3, the consumption growth rate associated with the target $\mathcal{X}$ is

$$\eta \left(A_K \left(1 - \mathcal{X}\right)^{\frac{1}{1-\sigma}} - \delta_o - \rho \right). \tag{36}$$

The comparison of (36) and (35) implies that a government interested in ensuring that workers' consumption-share remains at $\mathcal{X}$ prefers to tax capital rather than retard automation.[9]

5.4 Hours worked and the long-run labor share

Regardless of whether the labor share asymptotically vanishes, hours worked do. In this, our analysis is consistent with naïve predictions that neglect potentially countervailing effects stemming from complementarity between automated and non-automated tasks.

Although one might be tempted to conclude that the asymptotic vanishing of hours-worked makes it more likely that the labor share shrinks to zero, the reverse is in fact true. To see this,

[9] If the innovation that drives automation is endogenous and embedded in capital, then capital-taxation may additionally reduce θ. Our analysis has the benefit of cleanly separating this effect from those arising from endogenous factor returns.

consider briefly a perturbed version of our model, in which workers' labor is exogenously fixed at some interior level, $L_{w,t} \equiv \bar{L}_w \in (0,1)$, while capitalists' labor is exogenously set to zero, $L_{o,t} \equiv 0$.

Proposition 4 *If labor choices are exogenous and constant over time, the economy has a stable labor share if and only if*

$$\theta < \eta\,(1-\sigma)\,(A_K - \delta_o - \rho)\,. \tag{37}$$

Comparing Proposition 4 with our analysis above establishes that exogenous labor choices shrink the range of automation speeds associated with a stable labor share. Economically: Just as capital dominance is hindered by faster capital accumulation (see above), it is promoted by faster labor growth (i.e., labor that does not shrink towards 0).

5.5 r, g, and capitalist-worker inequality

Ceteris paribus, higher rates of return on capital favor capitalists at the expense of workers, a point emphasized by Piketty (2017). Here, we briefly discuss our analysis's implications for the asymptotic relation between the net return on capital, which we label $r = R - \delta_o$;[10] the growth rate of the economy, g_F; and capitalist-worker consumption inequality. From (14) and our equilibrium characterization: in a stable labor share equilibrium

$$r = \rho + \frac{\theta}{1-\sigma} \text{ and } \frac{r}{g_F} = \frac{\rho + \frac{\theta}{1-\sigma}}{\frac{\eta\theta}{1-\sigma}}$$

while in a capital dominant equilibrium

$$r = A_K - \delta_o \text{ and } \frac{r}{g_F} = \frac{A_K - \delta_o}{\eta\,(A_K - \delta_o - \rho)}.$$

On the one hand, faster rates of automation are associated both with higher values of r and with greater capitalist-worker consumption inequality, consistent with the partial equilibrium reasoning that higher rates of return on capital favor capitalists.

On the other hand, faster rates of automation are associated with lower ratios of r to g_F. The reason is simply that faster automation increases output growth proportionately more than it increases the return to capital. Combined with the fact that r exceeds g_F (as it must in any setting in which capital and output grow at the same rate, and the transversality condition (1) holds), it follows that while both r and g_F increase in the rate of automation θ, the ratio r/g_F decreases.

Consequently, the ratio r/g_F is negatively related to capitalist-worker inequality, at least asymptotically. This is true both as one varies the automation rate θ, and also as one moves across the different equilibria that co-exist in the case of strong complementarities ($\sigma + \eta < 1$).

[10] We evaluate r using the capitalists' $\delta_i = \delta_o$ as capitalists asymptotically hold all capital in all equilibria.

5.6 Observable trends

While we predominantly examine asymptotic factor shares in the full-automation limit, the key forces in our analysis also speak to three long-run empirical trends: (i) the decline in hours worked, (ii) the rising capital share, which our model naturally connects to a decline in TFP growth, and (iii) a reallocation in output shares towards services.

Time spent working converges to zero in all equilibria of our model, both for workers and capitalists. As such, our analysis predicts a long-term decline in hours worked, even away from the limit, consistent with empirical observation (Boppart and Krusell, 2020, and references therein). The prediction stems in part from the preference-specification with consumption and leisure as complements, $\eta < 1$, which does not belong to the class proposed by King et al. (1988) to generate stable hours worked despite consumption growth. Another driving force, which interacts with consumption-leisure complementarity, is task complementarity ($\sigma < 1$). As the economy accumulates capital but the per-capita time endowment stays fixed, labor-produced tasks become scarce and wages rise. That is, with task complementarity, automation is labor-augmenting (see also Aghion et al., 2019) and generates wage growth. With consumption-leisure complementarity, wage growth translates partially to leisure growth.

Regarding a rising capital share and falling TFP-growth, Lemma 1 implies that TFP growth in our economy is given by

$$g_{TFP} = \frac{\theta}{1-\sigma}\left(1 - \frac{1-X_t}{\alpha_t}\right). \tag{38}$$

Philippon (2023) argues that, empirically, TFP has grown linearly, which implies that g_{TFP} has dropped. The last term in (38) implies that our model features declining TFP growth if the capital share grows faster than the share of automated tasks, α_t. Observe first that the share of automated tasks grows at a rate $g_{\alpha,t} = \theta(1-\alpha_t)/\alpha_t$. It follows from equation (9) that (given $\sigma < 1$) the capital share grows faster than the share of automated tasks, and hence the growth rate of TFP declines, if and only if the marginal product of capital rises. US growth rates of capital, output, and the capital share from 1970-2019 satisfy this condition (see Table 1).

Lastly, our analysis has implications for the fraction of GDP stemming from each non-automated task or "sector" prior to its automation, namely, $X_t/(1-\alpha_t)$. Hence, the growth rate of each non-automated task's GDP share prior to automation is

$$g_{X,t} + \theta. \tag{39}$$

It follows that non-automated sectors grow faster than the overall economy whenever θ outweighs the rate of decline in the labor share at any point in time. (Note that, at least asymptotically, this condition is weaker than the condition for a stable labor share. That is: Even with capital dominance, the growth rate of as-yet non-automated sectors may exceed that of the overall economy.)

In the preliminary calibration we present in the following section, expression (39) is indeed positive for the US in the 1970-2019 period. It is natural to think of non-automated tasks largely as services, which Boppart (2014) shows have seen steadily rising expenditure shares in the US. Among the industries that have outgrown the overall economy at the fastest pace in recent decades, many naturally come to mind as examples of non-automated tasks, such as education, healthcare, restaurants, or performing arts. The industry with the biggest relative *decline* is manufacturing.[11]

6 A preliminary calibration

We make a first pass at calibrating our analysis, and in particular, assessing whether the economy is in the stable-labor share or the capital dominance region. By its nature, this exercise is highly speculative. But with that caveat, our calibration suggests that the economy will asymptote to a stable labor share.

Recall that whether or not a stable labor share equilibrium exists depends on whether the following inequality holds:

$$\frac{\theta}{1-\sigma} < A_K - \delta_o - \rho. \tag{40}$$

We consider the LHS and RHS of (40) in turn, starting with the LHS. Substitution of (8) into (10) and straightforward manipulation yields

$$\theta = (1-\sigma)(g_{F,t} - g_{L,t}) - \sigma g_{X,t}. \tag{41}$$

The growth rates $g_{X,t}$, $g_{L,t}$ and $g_{F,t}$ are observable. The parameter σ is the elasticity of substitution across tasks; as noted, it coincides with the production-based elasticity of substitution between capital and labor, and a significant literature is devoted to its estimation (e.g., Chirinko, 2008; Oberfield and Raval, 2021). Viewing the output aggregation into a single good through a consumption lens, the relevant elasticity of substitution is one across consumption goods (e.g., Nordhaus, 2021). Consequently, (41) links θ to observables and existing estimates of the elasticity σ.

Turning to the RHS of the key inequality (40), we obtain a straightforward lower bound for the parameter A_K by noting that the marginal product of capital satisfies[12]

$$A_K > F_{K,t} = \frac{1-X_t}{\frac{K_t}{F_t}}. \tag{42}$$

For inputs, we use the following from National Income Accounts (as of 2019), Huberman and Minns

[11] Using BEA data by industry *Manufacturing* lags total cumulative growth in value added between 1998 and 2021 by 32%, while *Food services* (25%), *Performing arts, spectator sports, and related activities* (25%), *Health care* (26%), and *Educational services* (31%) have all grown faster than total value added. Hubmer (2023) confirms quantitatively that these sectors—unlike manufacturing—have above-average labor shares.

[12] To see this formally, observe first that the assumption that all tasks that can be automated are indeed automated is that $\frac{A_K K_t}{\alpha_t} > \frac{A_L L_t}{1-\alpha_t}$. (As noted, this condition is satisfied once enough capital accumulation has occurred.) It then follows that $F_t < \frac{A_K K_t}{\alpha_t}$, and hence $F_{K,t} < A_K$.

(2007), and the US Census Bureau, all in per-capita terms (see Appendix B for details):

$$\left(g_F, g_L, g_K, g_X, g_{1-X}, X, \frac{K}{F}, \delta_o, \rho\right) = (1.81\%, -0.57\%, 1.44\%, -0.17\%, 0.28\%, 59.7\%, 3.63, 4.32\%, 2\%)$$

Our model abstracts from trends in the productivity parameters A_L and A_K; but we note that if instead labor-productivity A_L grows over time then expression (41) gives an upper bound for θ.

Figure 2 displays the rate of automation θ (calculated from (41)) and the key ratio $\frac{\theta}{1-\sigma}$ as a function of σ. Based on this speculative exercise, our analysis implies that the economy will asymptote to a stable labor share, as follows. First, the lower bound (42) for A_K implies

$$A_K - \delta_o - \rho > \frac{40.3\%}{3.63} - 4.32\% - 2\% = 4.79\%. \tag{43}$$

From Figure 2, the ratio $\frac{\theta}{1-\sigma}$ only approaches this bound if the elasticity parameter σ is close to 1 (it exceeds the bound for $\sigma \geq 0.94$), that is, outside the typical range of empirical estimates.[13] Allowing labor productivity, A_L, to rise over time would further reduce the estimate of θ.

Appendix B explores two alternative calibrations, both of which lead to the same conclusion of a stable labor share.

7 Conclusion

Asymptotically full automation does not necessarily spell doom for workers—provided that the speed of automation is not too fast. Conversely, workers are right to fear the consequences of faster rates of automation, which both lowers their consumption growth and leaves them with a negligible share of output. The threshold rate of automation separating these outcomes is closely related to observables, and a first-pass calibration suggests the current speed of automation is below this threshold. For faster rates of automation, our analysis suggests that workers are better served by redistribution funded by capital taxation rather than a deliberate slowing of automation.

[13] In his synthesis of the literature on capital-labor substitution in production, Chirinko (2008) writes that "the weight of the evidence suggests a value of σ in the range of 0.40–0.60." In a recent estimation for the manufacturing sector, Oberfield and Raval (2021) place it at 0.5–0.7. Given the dual role of σ in capturing both technology- and preference-based complementarities, we also note that estimates based on consumption expenditure similarly point towards gross complementarity (Nordhaus, 2021).

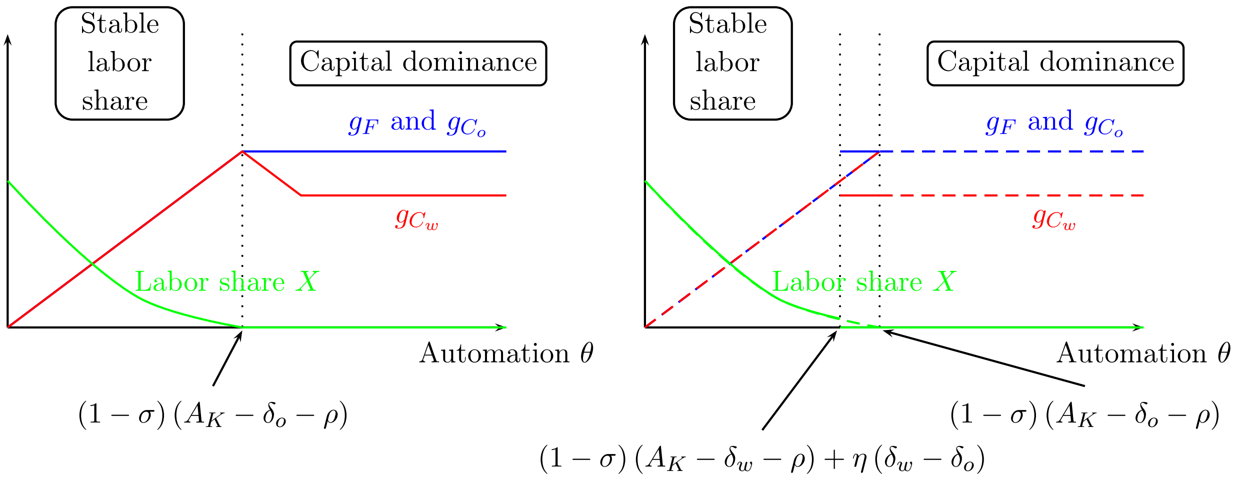

Figure 1: Asymptotic growth rates and labor share as a function of the automation speed θ. The left and right panels show the cases of weak complementarities ($\sigma+\eta>1$) and strong complementarities ($\sigma+\eta<1$). Dashed and solid lines correspond to multiple equilibria that arise for intermediate automation speeds.

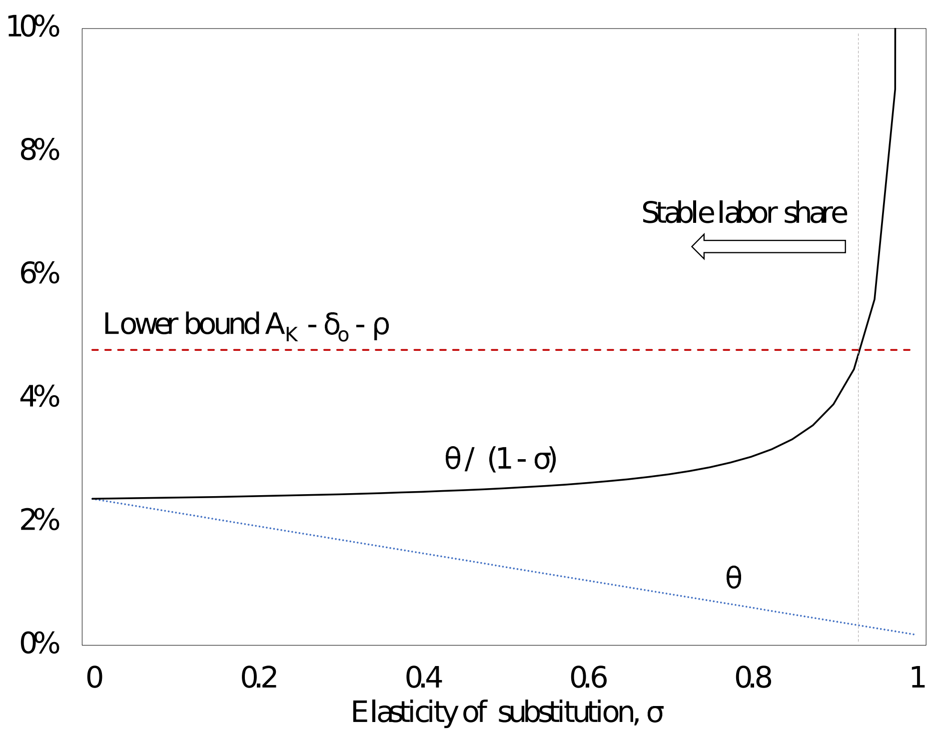

Figure 2: θ (dotted, blue) and $\frac{\theta}{1-\sigma}$ (solid, black) as functions of the elasticity of substitution between tasks, σ. The automation rate θ is inferred from (41). The dashed, red line marks the lower bound on $A_K-\delta_o-\rho$ from (43).

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A Proofs

Substituting Lemma 1 into the return and wage growth rates (7) and (8) gives

$$g_{R,t} = \frac{1}{\sigma}\left(X_t\left(g_{L,t}-g_{K,t}\right)+\frac{\theta}{1-\sigma}\left(1-\frac{1-X_t}{\alpha_t}\right)+\theta\frac{1-\alpha_t}{\alpha_t}\right) \tag{A-1}$$

$$g_{W,t} = \frac{1}{\sigma}\left(\left(1-X_t\right)\left(g_{K,t}-g_{L,t}\right)+\frac{\theta}{1-\sigma}\left(\sigma-\frac{1-X_t}{\alpha_t}\right)\right). \tag{A-2}$$

Output growth (12), $\alpha_t \to 1$, and (13) together imply that a stable labor share equilibrium exists only if

$$\lim\left(g_K - g_L\right) = \frac{\theta}{1-\sigma}. \tag{A-3}$$

Proof of Lemma 1: From the decomposition $F_t = K_t F_{K,t} + L_t F_{L,t}$:

$$\dot{F}_t = \dot{K}_t F_{K,t} + K_t \dot{F}_{K,t} + \dot{L}_t F_{L,t} + L_t \dot{F}_{L,t}$$

and hence (using also (8))

$$\begin{aligned}\frac{\dot{F}_t}{F_t} &= \frac{\dot{K}_t}{K_t}\frac{K_t F_{K,t}}{F_t}+\frac{K_t F_{K,t}}{F_t}\frac{\dot{F}_{K,t}}{F_{K,t}}+\frac{\dot{L}_t}{L_t}\frac{L_t F_{L,t}}{F_t}+\frac{L_t F_{L,t}}{F_t}\frac{\dot{F}_{L,t}}{F_{L,t}}\\ &= \left(\frac{\sigma-1}{\sigma}\right)\left(\left(1-X_t\right)\frac{\dot{K}_t}{K_t}+X_t\frac{\dot{L}_t}{L_t}\right)+\frac{1}{\sigma}\frac{\dot{F}_t}{F_t}+\frac{\theta}{\sigma}\left(\left(1-X_t\right)\frac{1-\alpha_t}{\alpha_t}-X_t\right),\end{aligned}$$

i.e.,

$$g_{F,t} = \left(1-X_t\right)g_{K,t}+X_t g_{L,t}+\frac{\theta}{\sigma-1}\left(\left(1-X_t\right)\frac{1-\alpha_t}{\alpha_t}-X_t\right),$$

which yields the result and completes the proof.

Proof of Lemma 2: First, there cannot be an equilibrium in which some group i both holds capital and has $\lim g_{1-L_i} < 0$, as follows. Suppose to the contrary that such an equilibrium exists. From the law of motion for capital,

$$\lim g_{K,i} = \bar{F}_K - \delta_i + \lim \frac{L_i F_L - C_i}{K_i}.$$

Since group i holds capital, its transversality condition can hold only if the final term on the RHS is non-positive, which in turn requires

$$\lim g_{C,i} \geq \lim g_{L,i} + \lim g_W = \lim g_W,$$

where the equality follows from the supposition that $\lim g_{1-L_i} < 0$. But since group i is not at the no-work corner, (21) and $\eta < 1$ imply that

$$\lim g_{C,i} = \eta \lim g_W + \lim g_{1-L_i} < \lim g_W,$$

contradicting the previous inequality and establishing the claim.

Second, there cannot be an equilibrium in which some group i does not hold capital and has $\lim g_{1-L_i} < 0$, as follows. Suppose to the contrary that such an equilibrium exists. Since by supposition $\lim g_{L_i} = 0$, the budget constraint for this non-capital-holding group i gives

$$\lim g_{C_i} = \lim g_W.$$

Substitution into (21) gives

$$\lim g_{1-L_i} = (1-\eta) \lim g_W,$$

and hence (by supposition, and $\eta < 1$)

$$\lim g_W = \lim g_{C_i} \leq 0.$$

From (10), $\lim X = 0$, and hence $\bar{F}_K = A_K$. Intertemporal optimality (25) and assumption (5) then imply

$$\lim g_{C_i} \geq \eta \left(\bar{F}_K - \delta_i - \rho \right) > 0,$$

contradicting $\lim g_{C_i} < 0$ and establishing the claim.

So far, we have established that $\lim g_{1-L_i} = 0$ for both groups. We now show that

$$\lim g_W > 0.$$

At least one group i must work (from the Inada condition for the marginal product of labor), and the intratemporal optimality condition (21) for this group gives

$$\lim g_{C_i} = \eta \lim g_W.$$

Suppose to the contrary that $\lim g_W \leq 0$. Then one obtains a contradiction exactly as above.

The consumption of both groups grows without bound, as follows. From the previous step, wages grow without bound; and also as above, at least one group must work. The combination of that group's intratemporal condition (21) and $\lim g_{1-L_i} = 0$ implies that the consumption of any group that works grows without bound. Moreover, if a group does not work, consumption of that group must grow even faster, completing the proof.

Proof of Lemma 3: We first show that workers supply strictly positive labor asymptotically. To see this, suppose to the contrary that workers do not work asymptotically. Hence workers hold capital, and capitalists work. From intermporal optimality (25),

$$\lim g_{C_w} = \eta\left(\bar{F}_K - \delta_w - \rho\right) < \eta\left(\bar{F}_K - \delta_o - \rho\right) \leq \lim g_{C_o}, \tag{A-4}$$

implying that workers work (since capitalists do), contradicting the original supposition.

Similarly, capitalists hold capital asymptotically. To see this, suppose to the contrary that capitalists do not hold capital. Hence workers hold capital, and capitalists work. Exactly the same steps as above imply (A-4), which contradicts the following implication of intratemporal optimality conditions:

$$\frac{1}{\eta}\lim g_{C_o} = \lim g_W \leq \frac{1}{\eta}\lim g_{C_w}.$$

Next, we show that capitalists do not work under capital dominance. Suppose to the contrary that capitalists and workers both work. By Corollary 1, workers do not hold capital. By capital dominance, aggregate labor income grows strictly slower than $\lim g_F = \lim g_K$, and hence workers' consumption C_w likewise grows strictly slower than $\lim g_K$. Capitalists' capital accumulation is given by

$$\frac{\dot{K}_{o,t}}{K_{o,t}} = F_{K,t} - \delta_o + \frac{L_{o,t}F_{L,t}}{F_t}\frac{F_t}{K_{o,t}} - \frac{C_{o,t}}{K_{o,t}}.$$

By capital dominance, the third term on the RHS converges to 0. The transversality condition for capitalists then implies that capitalists' consumption C_o asymptotically grows at the same rate as their capital holdings K_o, i.e.,

$$\lim g_{C_o} = \lim g_{K_o} = \lim g_K.$$

Hence capitalists' consumption grows strictly faster than workers' consumption, and the intratemporal optimality conditions imply that capitalists do not work, contradicting the supposition that they do.

Finally, we show that workers do not hold capital in stable labor share equilibrium. Suppose to the contrary that both capitalists and workers hold capital. (25) at equality for both groups directly implies $\lim g_{C_o} > \lim g_{C_w}$. Moreover, from Corollary 1, capitalists do not work, and the transversality condition for capitalists implies $\lim g_{K_o} = \lim g_{C_o}$. Workers' capital accumulation is given by

$$g_{K_w,t} = \frac{\dot{K}_{w,t}}{K_{w,t}} = F_{K,t} - \delta_w + \frac{L_{w,t}F_{L,t}}{F_t}\frac{F_t}{K_{w,t}} - \frac{C_{w,t}}{K_{w,t}}.$$

If $\lim g_{K_w} \geq \lim g_{K_o}$ then

$$\lim g_F = \lim g_K = \lim g_{K_w} \geq \lim g_{K_o} = \lim g_{C_o} > \lim g_{C_w}$$

implying

$$\lim g_{K_w} = \bar{F}_K - \delta_w + \lim X \lim \frac{F}{K} > \bar{F}_K - \delta_w,$$

violating the workers' transversality condition. If instead $\lim g_{K_w} < \lim g_{K_o}$ then

$$\lim g_F = \lim g_K = \lim g_{K_o} = \lim g_{C_o} > \lim g_{C_w},$$

implying that $\frac{L_{w,t} F_{L,t}}{F_t} F_t - C_{w,t}$ asymptotically grows at the same rate as aggregate capital K, which strictly exceeds the growth rate of worker capital K_w, implying $\lim g_{K_w} > \bar{F}_K - \delta_w$ and violating the workers' transversality condition. The contradiction completes the proof.

Proof of Lemma 4: Recall that $\lim g_F = \lim g_K$ (see (13)). From Corollary 2, the asymptotic growth rate of capitalists' consumption coincides with the the asymptotic growth rate of aggregate consumption, $\lim g_{C_o} = \lim g_C$. Asymptotically, aggregate consumption must grow weakly slower than output, $\lim g_C \leq \lim g_F$. For both groups i, the asymptotic growth rate of capital must be weakly below the asymptotic growth rate of consumption, $\lim g_{K_i} \leq \lim g_{C_i}$, since otherwise that group's transversality condition is violated.

We next show that $\lim g_{K_o} = \lim g_K$. If workers do not hold capital then this is immediate. If workers do hold capital, it suffices to show that $\lim g_{K_o} \geq \lim g_{K_w}$. Suppose to the contrary that $\lim g_{K_o} < \lim g_{K_w}$. In this case, capitalists do not work, and since the return on capital asymptotes to $\bar{F}_K$, capitalists' consumption must grow weakly slower that capitalists' capital holdings, $\lim g_{C_o} \leq \lim g_{K_o}$. Together, the above inequalities deliver

$$\lim g_{C_o} \leq \lim g_{K_o} < \lim g_{K_w} \leq \lim g_{C_w},$$

contradicting Corollary 2, and thereby establishing that $\lim g_{K_o} = \lim g_K$.

To complete the proof, simply note that

$$\lim g_{C_o} = \lim g_C \leq \lim g_F = \lim g_K = \lim g_{K_o} \leq \lim g_{C_o}.$$

establishing the result.

Proof of Proposition 1: We characterize the conditions for a capital-dominant equilibrium in which both groups hold capital to exist. From Lemma 3, workers work while capitalists do not . In a capital-dominant equilibrium, $\bar{F}_K = A_K$, and so from (25), the intertemporal conditions for capitalists and workers are

$$\begin{aligned} \lim g_{C_o} &= \eta \left(A_K - \delta_o - \rho \right) \\ \lim g_{C_w} &= \eta \left(A_K - \delta_w - \rho \right) \end{aligned}$$

while the intratemporal condition for workers is (using Lemma 2)

$$\lim g_W = \frac{1}{\eta} \lim g_{C_w} = A_K - \delta_w - \rho.$$

(Note that the above expression is positive by assumption (5).) Capital holdings grow according to

$$\begin{aligned}\lim g_{K_o} &= A_K - \delta_o - \lim \frac{C_o}{K_o} \\ \lim g_{K_w} &= A_K - \delta_w + \lim \frac{L_w F_L - C_w}{K_w},\end{aligned}$$

and from (A-2), wages grow according to

$$\lim g_W = \frac{1}{\sigma}\left(\lim g_K - \lim g_{L_w} - \theta\right).$$

Capitalists' transversality condition implies that C_o and K_o asymptotically grow at the same rate:

$$\lim g_{K_o} = \lim g_{C_o} = \eta\left(A_K - \delta_o - \rho\right).$$

We characterize an equilibrium in which C_w and K_w asymptotically grow at the same rate. In this case,

$$\lim g_{K_w} < \lim g_{K_o} = \lim g_K,$$

and so

$$\lim g_{L_w} = \eta\left(A_K - \delta_o - \rho\right) - \sigma\left(A_K - \delta_w - \rho\right) - \theta.$$

A worker's transversality condition is equivalent to

$$\lim g_{C_w} \geq \lim g_W + \lim g_{L_w}, \tag{A-5}$$

which substituting in the above expressions is equivalent to

$$\eta\left(A_K - \delta_w - \rho\right) \geq A_K - \delta_w - \rho + \eta\left(A_K - \delta_o - \rho\right) - \sigma\left(A_K - \delta_w - \rho\right) - \theta,$$

and hence to

$$\theta \geq (1-\sigma)\left(A_K - \delta_w - \rho\right) + \eta\left(\delta_w - \delta_o\right). \tag{A-6}$$

Note that $\lim g_{C_o} > \lim g_{C_w}$ together with the worker transversality condition (A-5) implies that the capital-dominance condition is satisfied; and also that capitalists indeed do not work. Moreover, the worker transversality condition implies that $\lim g_{L_w} < 0$.

Proof of Proposition 2: We characterize the conditions for a capital-dominant equilibrium in which workers do not hold capital to exist. By the similar arguments to those in the proof of Proposition 1, the asymptotic equilibrium conditions are as follows. (Relative to the proof of Lemma 1, the key difference is that workers' intertemporal optimality condition is replaced with an intratemporal budget constraint.)

$$\begin{aligned}
\lim g_{K_o} = \lim g_{C_o} &= \eta\left(A_K - \delta_o - \rho\right) \\
\lim g_W &= \frac{1}{\eta}\lim g_{C_w} \\
\lim g_{C_w} &= \lim g_W + \lim g_{L_w} \\
\lim g_W &= \frac{1}{\sigma}\left(\lim g_{K_o} - \lim g_{L_w} - \theta\right).
\end{aligned}$$

From a worker's intratemporal optimality and intratemporal budget constraint,

$$\lim g_{L_w} = (\eta - 1)\lim g_W.$$

Hence

$$\lim g_W = \frac{\lim g_{K_o} - \theta}{\sigma + \eta - 1}.$$

The capital-dominance condition is $\lim g_{K_o} > \lim g_W + \lim g_{L_w}$. Note that if the capital-dominance condition holds then $\lim g_{C_o} > \lim g_{C_w}$, which ensures that capitalists indeed do not work asymptotically. Substituting in, the capital-dominance condition is

$$\lim g_{K_o} > \eta\frac{\lim g_{K_o} - \theta}{\sigma + \eta - 1}.$$

The condition that workers asymptotically do not want to hold capital is (from (25), and substituting in for $\lim g_{C_w}$)

$$\lim g_W \geq A_K - \delta_w - \rho,$$

i.e.,

$$\lim g_W = \frac{\lim g_{K_o} - \theta}{\sigma + \eta - 1} \geq A_K - \delta_w - \rho = \frac{1}{\eta}\lim g_{K_o} - \left(\delta_w - \delta_o\right).$$

The above condition and (5) imply that $\lim g_W > 0$ and $\lim g_{L_w} < 0$.

Hence an equilibrium of this type exists if either $\sigma + \eta > 1$ and

$$\theta \in \left[\frac{1-\sigma}{\eta}\lim g_{K_o}, \frac{1-\sigma}{\eta}\lim g_{K_o} + (\sigma + \eta - 1)\left(\delta_w - \delta_o\right)\right]$$

or if $\sigma + \eta < 1$ and

$$\left[\frac{1-\sigma}{\eta}\lim g_{K_o} + (\sigma + \eta - 1)\left(\delta_w - \delta_o\right), \frac{1-\sigma}{\eta}\lim g_{K_o}.\right]$$

Substituting in for $\lim g_{K_o}$ yields the result.

Proof of Proposition 3: We characterize the conditions for a stable labor share equilibrium to exist. From Lemma 3, workers do not hold capital. Following similiar steps to those in the proofs of Propositions 1 and 2, but incorporating the possibility that capitalists work, the asymptotic equilibrium conditions are

$$
\begin{aligned}
\lim g_{C_o} &\geq \eta \lim g_W \\
\lim g_{C_o} &= \eta \left(\bar{F}_K - \delta_o - \rho\right) \\
\lim g_{K_o} &= \bar{F}_K - \delta_o - \lim \frac{C_o - F_L L_o}{K_o} \\
\lim g_W &= \frac{1}{\eta} \lim g_{C_w} \\
\lim g_{C_w} &= \lim g_W + \lim g_{L_w} \\
\lim g_W &= \frac{\theta}{1-\sigma}.
\end{aligned}
$$

From Lemma 4,

$$\lim g_F = \lim g_{K_o} = \lim g_{C_o} = \eta \left(\bar{F}_K - \delta_o - \rho\right).$$

We first show that aggregate labor growth matches worker-labor growth, i.e.,

$$\lim g_L = \lim g_{L_w}. \tag{A-7}$$

If capitalists do not work then (A-7) immediate. If instead capitalists work, note that capital evolves according to

$$\lim g_{K_o} = \bar{F}_K - \delta_o - \lim \frac{C_o - F_L L_o}{K_o}.$$

Capitalists' transversality constraint implies that their labor income grows weakly slower than the common growth rate of their consumption and capital. Moreover, if both capitalists and workers work, their consumption growth rates must asymptotically coincide (by Lemma 2 and the intratemporal optimality conditions). Hence

$$\lim g_W + \lim g_{L_o} \leq \lim g_{C_o} = \lim g_{C_w} = \lim g_W + g_{L_w}, \tag{A-8}$$

implying that $\lim g_{L_o} \leq \lim g_{L_w}$ and establishing (A-7).

From the workers' intratemporal optimality and intratemporal budget constraint,

$$\lim g_{L_w} = (\eta - 1) \lim g_W = (\eta - 1) \frac{\theta}{1-\sigma}.$$

Note that this condition ensures that $\lim g_{L_w} < 0$. Further, from (A-3), a stable labor share requires

$$\lim g_{K_o} - \lim g_L = \frac{\theta}{1-\sigma}.$$

From (A-7), it follows that

$$\lim g_{K_o} = \eta \frac{\theta}{1-\sigma},$$

which combined with capitalists' intertemporal optimality implies that the limiting rental rate is

$$\bar{F}_K = \frac{\theta}{1-\sigma} + \delta_o + \rho. \tag{A-9}$$

From (15), the asymptotic capital share is bounded away from one if and only if $\left(\frac{\bar{F}_K}{A_K}\right)^{1-\sigma} < 1$, which after substitution for $\bar{F}_K$ is equivalent to

$$\frac{\theta}{1-\sigma} + \delta_o + \rho < A_K.$$

Rearranging establishes the stable labor share condition, (32).

Workers' and capitalists' consumption grow at the same asymptotic rate, as follows. If capitalists do not work, this is immediate from the combination of definition of a stable labor share and the fact that output F, capital K_o and capitalist consumption C_o all grow at the same rate. If instead capitalists work, then it follows intratemporal optimality conditions, as already noted in (A-8).

Finally, the expression for the limiting labor share follows from the substitution of $\bar{F}_K$ into (15). This completes the proof.

Proof of Lemma 5: Equilibrium coexistence arises when complementarities are weak ($\sigma + \eta < 1$) and

$$\theta \in \left[(1-\sigma)\left(A_K - \delta_w + \rho\right) + \eta\left(\delta_w - \delta_o\right), (1-\sigma)\left(A_K - \delta_o + \rho\right)\right]. \tag{A-10}$$

In the stable labor share equilibrium,

$$\begin{aligned} \lim g_{C_o} = \lim g_{C_w} &= \frac{\eta}{1-\sigma}\theta \\ \lim g_{L_w} &= \frac{\eta - 1}{1-\sigma}\theta \end{aligned}$$

while in the capital-dominant equilibrium,

$$\begin{aligned} \lim g_{C_o} &= \eta\left(A_K - \delta_o + \rho\right) \\ \lim g_{C_w} &= \eta\left(A_K - \delta_w + \rho\right) \\ \lim g_{L_w} &= \eta\left(A_K - \delta_o - \rho\right) - \sigma\left(A_K - \delta_w - \rho\right) - \theta. \end{aligned}$$

It is immediate that $\lim g_{C_o}$ (respectively, $\lim g_{C_w}$) is higher (lower) in the capital dominant equilibrium than in the stable labor share equilibrium. Moreover, both comparisons are strict, with the exception of $\lim g_{C_o}$ at the upper boundary of the interval (A-10).

It remains to consider the labor growth rate $\lim g_{L_w}$. Because it is linear in θ in both equilibria, it suffices to consider the lower and upper boundaries of the interval (A-10).

At the lower end of the interval, in the stable labor share equilibrium

$$\lim g_{L_w} = (\eta - 1)(A_K - \delta_w - \rho) + \frac{\eta(\eta - 1)}{1 - \sigma}(\delta_w - \delta_o),$$

while in the capital-dominant equilibrium,

$$\lim g_{L_w} = (\eta - 1)(A_K - \delta_w - \rho),$$

which is strictly greater.

At the upper end of the interval, in the stable labor equilibrium

$$\lim g_{L_w} = (\eta - 1)(A_K - \delta_o - \rho),$$

while in the capital-dominant equilibrium,

$$\lim g_{L_w} = (\eta - 1)(A_K - \delta_o - \rho) + \sigma(\delta_w - \delta_o).$$

which again is strictly greater, completing the proof.

Proof of Corollary 3: The only case in which workers hold capital is characterized in Proposition 1. Workers' labor income grows at rate $g_{L_w} + g_W$, which evaluating equals

$$\eta(A_K - \delta_o - \rho) - \sigma(A_K - \delta_w - \rho) - \theta + (A_K - \delta_w - \rho).$$

Substituting in the equilibrium condition (27), the above expression is bounded above by

$$\eta(A_K - \delta_w - \rho),$$

which in turn equals the growth rate of worker's consumption, completing the proof.

Proof of Corollary 4: The result is immediate for strong complementarities ($\sigma + \eta < 1$), as covered in the main text. Here, we consider the case of weak complementarities ($\sigma + \eta > 1$). From Propositions 1 and 2, workers hold capital if and only if the automation rate θ exceeds the threshold value of

$$(1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) = (1 - \sigma)(A_K - \delta_o - \rho) + (\eta + \sigma - 1)(\delta_w - \delta_o). \quad \text{(A-11)}$$

Hence from Proposition 2, as θ approaches the threshold (A-11) from below, the growth rate of workers' consumption approaches

$$\eta \frac{\eta(A_K - \delta_o - \rho) - (1 - \sigma)(A_K - \delta_o - \rho) - (\eta + \sigma - 1)(\delta_w - \delta_o)}{\sigma + \eta - 1} = \eta(A_K - \delta_w - \rho),$$

which matches the growth rate of workers' consumption for any value of θ above the threshold (A-11). Hence (from Propositions 1 and 2 again), the growth rate of workers' consumption in a capital dominant equilibrium is simply

$$\max\left\{\eta\frac{\eta\left(A_K-\delta_o-\rho\right)-\theta}{\sigma+\eta-1},\eta\left(A_K-\delta_w-\rho\right)\right\},$$

where the first and second terms in the maximand correspond, respectively, to equilibria in which workers do not capital, and hold capital. The result is then immediate.

Proof of Proposition 4: We exogenously set labor choices to $L_{o,t}=0$ and $L_{w,t}\equiv\bar{L}_w\in(0,1)$.[14] Intertemporal optimality of capitalists, combined with the transversality condition, implies

$$\lim g_{K_o}=\eta\left(\bar{F}_K-\delta_o-\rho\right).$$

As before, aggregate capital growth equals capitalists' capital growth,

$$\lim g_K=\lim g_{K_o},$$

regardless of whether or not workers hold capital (since even if workers hold capital, their capital holdings grow more slowly than that of capitalists).

As before, the condition for capital dominance is that labor income asymptotically grows slower than capital income. Since labor supply is constant, this condition is simply

$$\lim g_W<\lim g_K.$$

From (A-2), capital dominance also requires

$$\lim g_W=\frac{1}{\sigma}\left(\lim g_K-\theta\right).$$

Finally, under capital dominance the return on capital asymptotes to A_K. Together, these observations imply that capital dominance requires

$$\theta>\eta(1-\sigma)(A_K-\delta_o-\rho). \tag{A-12}$$

Conversely, in a stable labor share equilibrium, consumption, capital income, and labor income must grow at the same rate. Combining (11)'s characterization of wage growth in a stable labor

[14] Note that we do not remove leisure from the agents' preferences. In the limit as consumption grows unbounded but leisure is bounded, the IES with consumption and leisure as gross complements tends to η rather than $1/\gamma$. Retaining this feature of preferences in the exogenous-labor case facilitates comparison with the endogenous-labor case.

share equilibrium with intertemporal optimality for capitalists gives

$$\frac{\theta}{1-\sigma} = g_W = \eta\left(\bar{F}_K - \delta_o - \rho\right).$$

As in the proof of Proposition 3, a stable labor share equilibrium requires $\bar{F}_K < A_K$, and hence requires

$$\theta < \eta(1-\sigma)(A_K - \delta_o - \rho),$$

which combined with (A-12) completes the proof.

B Calibration

B.1 Details for inputs

Table 1 below reports the inputs we use in our calibration, along with sources.[15,16]

Input	Description	Value	Source
g_F	Growth rate of output	2.79%	National income accounts
g_L	Growth rate of labor (per worker)	−0.57%	Huberman and Minns (2007)
g_K	Growth rate of capital	2.42%	National income accounts
g_X	Growth rate of labor share	−0.17%	National income accounts
g_{1-X}	Growth rate of capital share	0.28%	National income accounts
X	Labor share	59.7%	National income accounts
$\frac{K_t}{F_t}$	Capital/output ratio	3.63	National income accounts
δ_o	Depreciation	4.32%	National income accounts
ρ	Annual time preference	2%	Standard
	Population growth rate	0.98%	US Census Bureau

Table 1: Input values. Levels from national income accounts are estimated as of 2019. Growth rates refer to relative changes between 1970 and 2019. The growth rates of output and capital are for aggregate quantities; the calibration uses per-capita growth (subtracting population growth).

B.2 Alternative calibration approaches

We pursue two alternative calibration approaches complementing the exercise in Section 6.[17] The first one starts with an equation analogous to (41) but obtained from the law of motion of the capital share:

$$\frac{1-\alpha_t}{\alpha_t}\theta = \sigma g_{1-X,t} + (1-\sigma)\left(g_{K,t} - g_{F,t}\right). \tag{A-13}$$

Given observable growth rates for the capital share, capital, and output, the LHS can be estimated directly from existing estimates of the elasticity parameter σ.

To move from (A-13) to an estimate of θ one needs information about α_t, the fraction of tasks already automated. This number is hard to observe directly, and in our approach below we are agnostic about its value.

[15] The growth rates of output, labor supply, and the labor share are not constant in our model. We estimate $g_{F,t}$ and $g_{X,t}$ using data from 1970 to 2019. Our estimate for the growth rate of hours worked per capita comes from Huberman and Minns (2007).

[16] The empirical measurement of depreciation corresponds to $\frac{\lambda_o K_{o,t}}{K_t}\delta_o + \frac{\lambda_w K_{w,t}}{K_t}\delta_w \geq \delta_o$. Using a smaller value of δ_o than 4.32% would increase the estimated value of the RHS of (40), and reinforce the conclusion below that empirically the condition is likely to hold.

[17] A fourth possible approach to estimating θ would be to use the TFP equation (38). However, such an approach is susceptible to two significant pitfalls, and accordingly we do not pursue it. First, the inferred value of θ is very sensitive to the input α_t when α_t is close in value to the capital share of the economy X_t, which we cannot rule out a priori. Second, extracting θ from the TFP formula (38) is sensitive to the model assumption that technological advance consists solely of changes in the fraction of automated tasks.

For any given value of α_t, we can further tighten the estimate of A_K relative to the lower bound presented in (42). The expression for the capital share (14) can rewritten to yield

$$A_K = \frac{F_t}{K_t}\left(\frac{\alpha_t}{(1-X_t)^\sigma}\right)^{\frac{1}{1-\sigma}}. \tag{A-14}$$

The drawback of this expression, relative to the lower bound in (42) is that it requires an assumption on α_t. Note, however, that both the value of θ inferred from (A-13) and the value of A_K inferred from expression (A-14) are increasing in the automation share α_t, and hence both the estimated LHS and RHS of the key inequality (40) are likewise increasing in α_t.

Using once again inputs from Table 1, with $g_{K,t}$ and $g_{1-X,t}$ estimated over the same 1970–2019 sample as $g_{F,t}$, Table 2 displays, for a range of possible values of α_t and σ, the rate of automation θ (calculated from (A-13)), the key ratio $\frac{\theta}{1-\sigma}$, and the productivity parameter A_K (calculated using (A-14)). The ratio $\frac{\theta}{1-\sigma}$ only exceeds this bound if the elasticity parameter σ is relatively close to 1 *and* the fraction of tasks already automated (α_t) is high. Note that the baseline calibration in Section 6 implies an estimate of α_t (from equations (41) and (A-13) and the observable growth rates). These estimates indicate that the majority of tasks is already automated but are *decreasing* in σ: for $\sigma = 0.8$, the implied α_t is 0.8, dropping to 0.6 for $\sigma = 0.9$.

In the table, we use color shading to highlight the combinations of σ and α_t for which the ratio $\frac{\theta}{1-\sigma}$ either exceeds 4.79%, or at least approaches it. But those parameter choices that deliver $\frac{\theta}{1-\sigma}$ anywhere close to the boundary of 4.79% *also* imply large values for A_K, and hence for $A_K - \delta_o - \rho$, so that the stable-labor share inequality (40) continues to hold.[18]

Finally, an alternative and independent approach to estimating $\frac{1-\alpha_t}{\alpha_t}\theta$ is as follows. The fraction of investment devoted to new automation, ϕ_t, equals

$$\phi_t = \frac{\dot{\alpha}_t \frac{K_t}{\alpha_t}}{\dot{K}_t} = \frac{\frac{1-\alpha_t}{\alpha_t}\theta}{g_{K,t}}. \tag{A-15}$$

Rearranging:

$$\frac{1-\alpha_t}{\alpha_t}\theta = \phi_t g_K. \tag{A-16}$$

Table 2 shows the results of inferring θ from (A-16) instead of from (A-13), for a range of values of the fraction of investment devoted to new automation.The conclusions are the same as those drawn from Figure 2 and Table 2.

[18]Note that high values of α_t lead to extremely high estimates of A_K, the productivity of capital in an all-capital economy. The reason is as follows. First, note from (14) that the capital share is decreasing in the amount of "effective" capital $A_K K_t$, since tasks are complements ($\sigma < 1$). The current capital share in the economy is much less than 100%. If one believes that most tasks are already automated, the only way to explain the observed capital share is to posit that there is a large amount of "effective" capital $A_K K_t$. Given observed levels of capital K_t, this in turn implies that A_K must be high.

	$\sigma = 0.6$			$\sigma = 0.8$			$\sigma = 0.9$		
α_t	θ	$\theta/(1-\sigma)$	A_K	θ	$\theta/(1-\sigma)$	A_K	θ	$\theta/(1-\sigma)$	A_K
0.1	0.00%	0.01%	0.33%	0.02%	0.09%	0.01%	0.02%	0.24%	0.00%
0.2	0.01%	0.02%	1.89%	0.04%	0.19%	0.33%	0.05%	0.54%	0.01%
0.3	0.01%	0.03%	5.20%	0.07%	0.33%	2.49%	0.09%	0.93%	0.57%
0.4	0.02%	0.04%	10.68%	0.10%	0.51%	10.48%	0.15%	1.45%	10.10%
0.5	0.02%	0.06%	18.66%	0.15%	0.77%	31.99%	0.22%	2.18%	94.04%
0.6	0.04%	0.09%	29.43%	0.23%	1.15%	79.60%	0.33%	3.27%	582.28%
0.7	0.06%	0.14%	43.27%	0.36%	1.79%	172.04%	0.51%	5.08%	2720.19%
0.8	0.10%	0.24%	60.41%	0.61%	3.06%	335.42%	0.87%	8.71%	10339.94%
0.9	0.22%	0.54%	81.10%	1.38%	6.89%	604.44%	1.96%	19.60%	33577.12%

Table 2: $\frac{\theta}{1-\sigma}$ and A_K as functions of the current level of automation, α_t, and the elasticity of substitution between tasks, σ. The automation rate θ is inferred from (A-13). Color shading highlights values of σ and α_t for which the ratio $\frac{\theta}{1-\sigma}$ approaches or exceeds the lower bound (43).

	$\phi_t = 5\%$		$\phi_t = 15\%$		$\phi_t = 25\%$		
α_t	θ	$\theta/(1-\sigma)$	θ	$\theta/(1-\sigma)$	θ	$\theta/(1-\sigma)$	A_K
0.1	0.01%	0.02%	0.02%	0.12%	0.04%	0.40%	0.33%
0.2	0.02%	0.04%	0.05%	0.27%	0.09%	0.90%	1.89%
0.3	0.03%	0.08%	0.09%	0.46%	0.15%	1.54%	5.20%
0.4	0.05%	0.12%	0.14%	0.72%	0.24%	2.40%	10.68%
0.5	0.07%	0.18%	0.22%	1.08%	0.36%	3.60%	18.66%
0.6	0.11%	0.27%	0.32%	1.62%	0.54%	5.40%	29.43%
0.7	0.17%	0.42%	0.50%	2.52%	0.84%	8.40%	43.27%
0.8	0.29%	0.72%	0.86%	4.32%	1.44%	14.40%	60.41%
0.9	0.65%	1.62%	1.94%	9.72%	3.24%	32.40%	81.10%

Table 3: $\frac{\theta}{1-\sigma}$ and A_K as functions of the current level of automation, α_t, and the fraction of investment devoted to new automation (either 10%, 20%, or 30%). The automation rate θ is inferred from (A-16). The table uses $\sigma = 0.6$ throughout; adopting higher values of σ only strengthens the conclusion that (40) holds. Color shading is as in Table 2.

C Analysis of representative agent case

We consider the representative agent case, i.e., $\delta_o = \delta_w$. We simply write δ for this common value, and related, drop all group-specific subscripts.

By (13), capital and output asymptotically grow at the same rate. Moreover, consumption must asymptotically grow at this same rate, as follows. Certainly consumption cannot asymptotically grow faster than F. Since F and K asymptotically grow at the same rate, this in turn implies that C cannot asymptotically grow faster than K. But nor can C asymptotically grow slower than K; if it did, $\frac{C}{K} \to 0$, and so

$$g_K \to \frac{F}{K} - \delta \geq F_K - \delta,$$

which would violate the transversality condition. Hence F, K, and C must all grow at the same rate asymptotically,

$$\lim g_F = \lim g_K = \lim g_C. \tag{A-17}$$

Inada conditions in the production function imply that the representative agent both works and holds capital, and so intra- and intertemporal optimality implies

$$g_C - g_{1-L} = \eta g_W, \tag{A-18}$$

$$\lim g_C = \eta \left(\bar{F}_K - \delta - \rho \right). \tag{A-19}$$

Lemma A-1 *In the representative agent benchmark, an equilibrium with a stable labor share exists if*

$$\theta < (1-\sigma)(A_K - \delta - \rho). \tag{A-20}$$

The asymptotic growth rate of output, capital, and consumption is

$$g_F = g_K = g_C = \frac{\eta\theta}{1-\sigma}. \tag{A-21}$$

Wages grow faster than consumption

$$\lim g_W = \frac{g_C}{\eta}, \tag{A-22}$$

while labor converges towards 0 *according to*

$$\lim g_L = \left(1 - \frac{1}{\eta}\right) g_C < 0. \tag{A-23}$$

The labor share converges towards

$$\lim X = 1 - \left(\frac{\delta + \rho + \frac{\theta}{1-\sigma}}{A_K} \right)^{1-\sigma} \tag{A-24}$$

Lemma A-2 *In the representative agent benchmark, an equilibrium with capital dominance exists* *if*

$$\theta > (1-\sigma)(A_K - \delta - \rho). \tag{A-25}$$

The asymptotic growth rate of output, capital, and consumption is

$$g_F = g_K = g_C == \eta (A_K - \delta - \rho). \tag{A-26}$$

Wages grow faster than consumption,

$$\lim \frac{\dot{F}_L}{F_L} = \frac{g_C}{\eta}, \tag{A-27}$$

while labor converges towards 0 *according to*

$$\lim g_L = \left(1 - \frac{\sigma}{\eta}\right) g_C - \theta < 0. \tag{A-28}$$

Proof of Lemma A-1: Recall that a stable labor share arises if wages asymptotically grow according to (11), and asymptotic capital and labor growth are linked via (A-3). From (11), wages grow without bound,

$$F_L \to \infty. \tag{A-29}$$

Moreover, the asymptotic growth rate of leisure must be zero,

$$\lim g_{1-L} = 0, \tag{A-30}$$

as follows. If instead $\lim g_{1-L} < 0$ then intratemporal optimality (A-18) and the complementarity of labor and leisure ($\eta < 1$) implies that $\lim g_W > \lim g_C$. But $\lim g_{1-L} < 0$ also implies that $\lim g_L = 0$, and hence (11) and (A-3) imply that $\lim g_W = g_K$, a contradiction (since $g_K = g_C$ by (A-17)).

So intratemporal optimality (A-18) implies that wages grow faster than consumption,

$$\lim g_W = \frac{1}{\eta} \lim g_C. \tag{A-31}$$

Substituting in (11) gives

$$\lim g_C = \frac{\eta\theta}{1-\sigma}. \tag{A-32}$$

Substituting into intertemporal optimality (A-19) gives

$$\bar{F}_K = \delta + \rho + \frac{\theta}{1-\sigma}. \tag{A-33}$$

The condition for a stable labor share is simply $\bar{F}_K < A_K$, i.e.,

$$\theta < (1-\sigma)(A_K - \delta - \rho). \tag{A-34}$$

The growth rate of labor is given by (A-3),

$$\lim g_L = \left(1 - \frac{1}{\eta}\right) g_C < 0. \tag{A-35}$$

The consumption to capital ratio $\lim \frac{C}{K}$ is determined by the law of motion for capital: again using $g_C = g_K$ and intertemporal optimality (A-19),

$$\lim \frac{C}{K} = \frac{\bar{F}_K}{\lim \frac{KF_K}{F}} - \delta - g_K = \bar{F}_K^{\sigma} A_K^{1-\sigma} - \delta - \eta \left(\bar{F}_K - \delta - \rho\right), \tag{A-36}$$

which is strictly positive since $\eta < 1$ and $A_K > \bar{F}_K > \delta$, completing the proof.

Proof of Lemma A-2: Under capital dominance, (16) holds. From (A-2), wages grow according to

$$\lim g_W = \frac{1}{\sigma} \left(\lim g_K - \lim g_L - \theta\right). \tag{A-37}$$

The capital dominance condition is

$$\lim g_W < g_F - \lim g_L. \tag{A-38}$$

The asymptotic growth rate of leisure must be zero,

$$\lim g_{1-L} = 0, \tag{A-39}$$

as follows. The intratemporal optimality (A-18) condition implies $\lim g_C \leq \eta \lim g_W$. If $\lim g_{1-L} < 0$ then $\lim g_L = 0$, and the capital dominance condition reduces to $\lim g_W < \lim g_F = \lim g_C$. Since $\eta < 1$, these two bounds on g contradict each other.

So intratemporal optimality (A-18) implies that wages grow faster than consumption,

$$\lim g_W = \frac{g_C}{\eta}. \tag{A-40}$$

In particular, (11) implies that wages grow without bound,

$$F_L \to \infty. \tag{A-41}$$

Substituting into intertemporal optimality (A-19) gives consumption growth in terms of the return on capital, which under complete automation is simply A_K:

$$\lim g_C = \eta \left(A_K - \delta - \rho\right). \tag{A-42}$$

Combining the two expressions above for the growth rate of wages, and using $g_K = g_C$, the growth rate of labor equals

$$\lim g_L = \left(1 - \frac{\sigma}{\eta}\right) g_C - \theta. \tag{A-43}$$

Note that the capital dominance condition and the expression for the growth rate of wages directly imply that

$$\lim g_L < 0.$$

The capital dominance condition rewrites as

$$\frac{g_C}{\eta} < \frac{\sigma}{\eta} g_C + \theta, \tag{A-44}$$

i.e.,

$$\theta > (1 - \sigma)\left(A_K - \delta - \rho\right). \tag{A-45}$$

The consumption to capital ratio $\lim \frac{C}{K}$ is determined by the law of motion for capital,

$$\lim \frac{C}{K} = A_K - \delta - g, \tag{A-46}$$

which is strictly positive since $\eta < 1$ and $A_K > \delta$, completing the proof.